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ON CERTAIN TWO-DIMENSIONAL APPLICATIONS OF THE COUPLE STRESS THEORY

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Abstract—The two-dimensional problem of equilibrium of a perfectly elastic body with couple-stresses is considered. The general form of solution of the problem in the case of an orthotropic medium is given in Section 1. Section 2 shows that there exists a close analogy between the equations governing the behavior of a plane rectangular lattice composed of rigidly interconnected elastic beams, and the general set of equations of the two-dimensional couple-stress theory for certain orthotropic bodies. Section 3 contains two problems which demonstrate the effects of couple-stresses upon the behavior of stresses and displacements in the vicinity of a circular inclusion and a circular hole in an infinite elastic plane.

INTRODUCTION

THE two-dimensional problem of equilibrium of a perfectly elastic orthotropic body with couple-stresses is considered. By following the approach suggested by Koiter [1] it can be assumed that the simplest physical model of such a body is characterized, in the case of isotropy, by four independent elastic constants E, G, l and η. E and G are the usual Young's and shear moduli, l and η are two new constants arising through the introduction of couple stresses. l has the dimension of length and appears to be of microscopic magnitude for most real materials. The cross-sensitivity constant η is dimensionless and has the value between $+1$ and -1. If the same general approach is followed for the orthotropic case the number of independent elastic constants should reach $9+12 = 21$. When considering either the case of plane stress or plane strain without specifying the elastic constants associated with the z direction, the number of independent coefficients reduces to six. In the usual engineering notation these constants can be written E_x, E_y, G, $E_{xy} = E_y/\nu_y = E_x/\nu_x$, l_x and l_y.

Section 2 of the paper shows that there exists a close analogy between the equations governing the behavior of a plane rectangular lattice, composed of rigidly interconnected elastic beams loaded on the boundary, and the general set of equations (derived in Section 1) of the two-dimensional couple-stress theory for certain orthotropic bodies. This analogy also adds clarity to the physical meaning of the two characteristic lengths l_x and l_y.

Section 3 contains two problems which demonstrate the effects of couple-stresses. The first problem involves investigating the stress concentration around a rigid circular inclusion in an infinite elastic plate subject to simple compression in one direction. The

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second problem is concerned with the displacement and stress distribution around a circular hole in an infinite elastic plate when the boundary of the hole is subject to a torque. In both problems the influence of couple-stresses is quite significant, though limited to the immediate vicinity of the inclusion or hole. A more detailed discussion of these problems can be found in the unpublished paper [6].

1. PLANE COUPLE-STRESS PROBLEM

Consider the state of equilibrium in a perfectly elastic body with couple-stresses. The general set of equations governing the problem for the case of vanishing body forces and body couples, as given by Koiter [1], can be reduced to

(a) three equations of equilibrium

$$s_{mn,m} - \tfrac{1}{2}\varepsilon_{imn}m_{ji,jm} = 0; \tag{1.1}$$

(b) fifteen strain–displacement relations

$$\gamma_{ij} = u_{(i,j)}, \qquad \varkappa_{ij} = \omega_{j,i}; \tag{1.2}$$

(c) fifteen stress–strain relations

$$s_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \qquad m_{ij} = \frac{\partial W}{\partial \varkappa_{ij}}; \tag{1.3}$$

(d) five boundary conditions for each point of the boundary, expressed in either stresses or displacements.

The following notation (basically the same as found in Ref. [1]) is introduced here.

$s_{ij} = \sigma_{(i,j)}$—the symmetric part of the generally non-symmetric force-stress tensor σ_{ij};

$m_{ij} = \mu_{ij} - \frac{1}{3}\mu_{kk}\delta_{ij}$—deviatoric part of the couple-stress tensor μ_{ij};

u_i—components of the displacement vector;

γ_{ij}—components of the strain tensor;

$\omega_i = \frac{1}{2}\varepsilon_{ijk}u_{k,j}$—components of the rotation vector;

$\varkappa_{ij}$—components of the torsion-flexure tensor;

$W = W(\gamma_{ij}, \varkappa_{ij})$—strain energy function (homogeneous quadratic function of the deformation variables γ_{ij} and $\varkappa_{ij}$);

ε_{ijk}—permutation symbol and δ_{ij}—Kronecker delta.

The system of equations (1.1), (1.2) and (1.3), in the particular case of a two-dimensional problem, reduces (see Ref. [2] by Mindlin) to the following set of equations.

(a) Two equations of equilibrium

$$\begin{aligned} s_{xx,x} + s_{xy,y} + \tfrac{1}{2}(m_{xz,x} + m_{yz,y})_{,y} &= 0, \\ s_{xy,x} + s_{yy,y} - \tfrac{1}{2}(m_{xz,x} + m_{yz,y})_{,x} &= 0; \end{aligned} \tag{1.4}$$

(b) three compatibility conditions

$$\begin{aligned} \varkappa_{xz} &= \gamma_{xy,x} - \gamma_{xx,y}, \\ \varkappa_{yz} &= -\gamma_{xy,y} + \gamma_{yy,x}, \\ \varkappa_{xz,y} &= \varkappa_{yz,x}; \end{aligned} \tag{1.5}$$

(c) Five stress–strain relations obtained as a result of the assumption that the strain energy function has the form

$$W = \frac{1}{2}\left(\frac{E_x}{1-\nu_x\nu_y}\gamma_{xx}^2 + \frac{E_y}{1-\nu_x\nu_y}\gamma_{yy}^2\right) + \frac{E_x\nu_y}{1-\nu_x\nu_y}\gamma_{xx}\gamma_{yy}$$
$$+G\gamma_{xy}^2 + 2Gl_y^2\varkappa_{yz}^2 + 2Gl_x^2\varkappa_{xz}^2.$$

If, in addition, the assumption $\nu_x = \nu_y = 0$ is made, then

$$\begin{aligned} s_{xx} &= E_x\gamma_{xx}, & m_{xz} &= 4Gl_x^2\varkappa_{xz}, \\ s_{yy} &= E_y\gamma_{yy}, & m_{yz} &= 4Gl_y^2\varkappa_{yz}, \\ s_{xy} &= 2G\gamma_{xy}, \end{aligned} \tag{1.6}$$

hence the plane state of stress and plane state of strain become equivalent.

If the notation $\eta_x = 2G/E_x, \eta_y = 2G/E_y$ ($\eta_x = \eta_y = 1$ for the isotropic case) is introduced the stress compatibility conditions, obtained by substituting equations (1.6) into (1.5), can be written

$$\begin{aligned} m_{xz} &= 2l_x^2(s_{xy,x} - \eta_x s_{xx,y}), \\ m_{yz} &= 2l_y^2(-s_{xy,y} + \eta_y s_{yy,x}), \\ l_y^2 m_{xz,y} &= l_x^2 m_{yz,x}. \end{aligned} \tag{1.7}$$

The couple-stresses (m_{xz} and m_{yz}) can be eliminated from the compatibility and equilibrium equations by substituting from the first two equations (1.7) into equations (1.6) and the third of equations (1.7)

$$\begin{aligned} \frac{\partial}{\partial x}\left(1 - l_x^2\eta_x\frac{\partial^2}{\partial y^2}\right)s_{xx} + \eta_y l_y^2\frac{\partial^3 s_{yy}}{\partial x\partial y^2} + \frac{\partial}{\partial y}\left(1 + l_x^2\frac{\partial^2}{\partial x^2} - l_y^2\frac{\partial^2}{\partial y^2}\right)s_{xy} &= 0, \\ \eta_x l_x^2\frac{\partial^3 s_{xx}}{\partial x^2\partial y} + \frac{\partial}{\partial y}\left(1 - \eta_y l_y^2\frac{\partial^2}{\partial x^2}\right)s_{yy} + \frac{\partial}{\partial x}\left(1 - l_x^2\frac{\partial^2}{\partial x^2} + l_y^2\frac{\partial^2}{\partial y^2}\right)s_{xy} &= 0, \\ \eta_x\frac{\partial^2 s_{xx}}{\partial y^2} + \eta_y\frac{\partial^2 s_{yy}}{\partial x^2} - 2\frac{\partial^2 s_{xy}}{\partial x\partial y} &= 0. \end{aligned} \tag{1.8}$$

The general solution of equations (1.8) can be expressed in terms of three stress potentials U_i ($i = 1, 2, 3$)

$$\begin{aligned} s_{xx} &= \frac{\partial}{\partial x}\left[-L_{xy}U_1 + \eta_y\frac{\partial^2(1+\Omega)}{\partial x\partial y}U_2\right] - M_{xy}U_3, \\ s_{yy} &= \frac{\partial}{\partial y}\left[\eta_x\frac{\partial^2(1+\Omega)}{\partial x\partial y}U_1 - L_{yx}U_2\right] - M_{yx}U_3, \\ s_{xy} &= N_{xy}U_1 + N_{yx}U_2 + \frac{\partial^2}{\partial x\partial y}\left(1 - \eta_y l_y^2\frac{\partial^2}{\partial x^2} - \eta_x l_x^2\frac{\partial^2}{\partial y^2}\right)U_3, \end{aligned} \tag{1.9}$$

$$m_{xz} = 2l_x^2\left[\frac{\partial}{\partial x}(\Omega_{xy}U_1 - \Omega_{yx}U_2) + Q_{xy}U_3\right],$$

$$m_{yz} = 2l_y^2\left[\frac{\partial}{\partial y}(\Omega_{xy}U_1 - \Omega_{yx}U_2) - Q_{yx}U_3\right],$$

in which the differential operators Ω, Ω_{xy}, L_{xy}, M_{xy}, N_{xy} and Q_{xy} are defined as follows:

$$\Omega = l_x^2\frac{\partial^2}{\partial x^2} + l_y^2\frac{\partial^2}{\partial y^2}, \qquad \Omega_{xy} = \eta_x\frac{\partial}{\partial y}\left(\eta_y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

$$L_{xy} = \eta_y\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial y^2} - \eta_y\frac{\partial^2\Omega}{\partial x^2},$$

$$M_{xy} = \frac{\partial^2}{\partial y^2}\left[1 - \Omega + 2(l_x^2 - \eta_y l_y^2)\frac{\partial^2}{\partial x^2}\right], \tag{1.10}$$

$$N_{xy} = \eta_x\frac{\partial}{\partial y}\left[\eta_y\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right],$$

$$Q_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial^2}{\partial x^2} + \eta_x\frac{\partial^2}{\partial y^2} - l_y^2\left(\eta_y\frac{\partial^4}{\partial x^4} + 2\eta_x\eta_y\frac{\partial^4}{\partial x^2\partial y^2} + \eta_x\frac{\partial^4}{\partial y^4}\right)\right].$$

The remaining differential operators can be obtained from (1.10) by interchanging the variables x and y. When equations (1.9) and (1.10) are substituted into (1.8) the governing equations for the determination of the stress-potentials become

$$\eta_y l_x^2\frac{\partial^6 U_i}{\partial x^6} + \eta_y(l_y^2 + 2\eta_x l_x^2)\frac{\partial^6 U_i}{\partial x^4\partial y^2} + \eta_x(l_x^2 + 2\eta_y l_y^2)\frac{\partial^6 U_i}{\partial x^2\partial y^4}$$
$$+\eta_x l_y^2\frac{\partial^6 U_i}{\partial y^6} - \left(\eta_y\frac{\partial^4 U_i}{\partial x^4} + 2\frac{\partial^4 U_i}{\partial x^2\partial y^2} + \eta_x\frac{\partial^4 U_i}{\partial y^4}\right) = 0. \tag{1.11}$$

In certain cases a simplified choice of two stress functions Φ and Ψ is more convenient. For example, if it is assumed that

$$s_{xx} = \Phi_{,yy} - \frac{l_x}{l_y}\Psi_{,xy}, \qquad s_{yy} = \Phi_{,xx} + \frac{l_y}{l_x}\Psi_{,xy},$$

$$s_{xy} = -\Phi_{,xy} + \frac{1}{2}\left(\frac{l_x}{l_y}\Psi_{,xx} - \frac{l_y}{l_x}\Psi_{,yy}\right), \tag{1.12}$$

$$m_{xz} = \frac{l_x}{l_y}\Psi_{,x}, \qquad m_{yz} = \frac{l_y}{l_x}\Psi_{,y}$$

it is easily verified, by simple substitution, that equations (1.12) satisfy the required conditions (1.4) and (1.7) provided that Φ and Ψ are solutions of the following two equations

$$2l_x l_y\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2} + \eta_x\frac{\partial^2}{\partial y^2}\right)\Phi + \frac{\partial}{\partial x}\left[1 + (l_y^2 - 2\eta_x l_x^2)\frac{\partial^2}{\partial y^2} - l_x^2\frac{\partial^2}{\partial x^2}\right]\Psi = 0,$$
$$2l_x l_y\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2} + \eta_y\frac{\partial^2}{\partial x^2}\right)\Phi - \frac{\partial}{\partial y}\left[1 + (l_x^2 - 2\eta_y l_y^2)\frac{\partial^2}{\partial x^2} - l_y^2\frac{\partial^2}{\partial y^2}\right]\Psi = 0. \tag{1.13}$$

When equations (1.13) are solved simultaneously to separate Φ and Ψ, equations (1.11) are once more obtained. Thus, while equations (1.13) are sufficient for the determination of Φ and Ψ, the necessary conditions are equations (1.11).

For the isotropic case $E_x = E_y = E$, $l_x = l_y = l$, $E = 2G$ and the stresses, in terms of the potentials, can be written

$$\begin{aligned}
s_{xx} &= -\frac{\partial}{\partial x}\left[\left(1-l^2\frac{\partial^2}{\partial x^2}\right)\nabla^2+\frac{\partial^2}{\partial y^2}\right]U_1+\frac{\partial^3}{\partial x^2\partial y}(1+l^2\nabla^2)U_2-\frac{\partial^2}{\partial y^2}(1-l^2\nabla^2)U_3,\\
s_{yy} &= \frac{\partial^3}{\partial x\partial y^2}(1+l^2\nabla^2)U_1-\frac{\partial}{\partial y}\left[\left(1-l^2\frac{\partial^2}{\partial y^2}\right)\nabla^2+\frac{\partial^2}{\partial x^2}\right]U_2-\frac{\partial^2}{\partial x^2}(1-l^2\nabla^2)U_3,\\
s_{xy} &= \frac{\partial}{\partial y}\left(l^2\frac{\partial^2\nabla^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)U_1+\frac{\partial}{\partial x}\left(l^2\frac{\partial^2\nabla^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)U_2+\frac{\partial^2}{\partial x\partial y}(1-l^2\nabla^2)U_3,\\
m_{xz} &= 2l^2\left[\frac{\partial}{\partial x}\left(\frac{\partial\nabla^2U_1}{\partial y}-\frac{\partial\nabla^2U_2}{\partial x}\right)+\frac{\partial}{\partial y}(\nabla^2-l^2\nabla^4)U_3\right],\\
m_{yz} &= 2l^2\left[\frac{\partial}{\partial y}\left(\frac{\partial\nabla^2U_1}{\partial y}-\frac{\partial\nabla^2U_2}{\partial x}\right)-\frac{\partial}{\partial x}(\nabla^2-l^2\nabla^4)U_3\right],
\end{aligned}\tag{1.14}$$

and the equations governing the potentials become

$$l^2\nabla^6U_i-\nabla^4U_i = 0. \tag{1.15}$$

A simplified version of the stress potentials for the isotropic case, originally introduced by Mindlin, is obtained by substituting

$$\Phi = (1-l^2\nabla^2)U_3, \qquad \Psi = (1+l^2\nabla^2)(U_{2,x}-U_{1,y}) \tag{1.16}$$

into equation (1.14),

$$\begin{aligned}
s_{xx} &= \Phi_{,yy}-\Psi_{,xy}, \qquad s_{yy} = \Phi_{,xx}+\Psi_{,xy},\\
s_{xy} &= -\Phi_{,xy}+\tfrac{1}{2}(\Psi_{,xx}-\Psi_{,yy}),\\
m_{xz} &= \Psi_{,x}, \qquad m_{yz} = \Psi_{,y}.
\end{aligned}\tag{1.17}$$

Similarly the equations for Φ and Ψ become

$$\begin{aligned}
2l^2\nabla^2\Phi_{,y} &= -(\Psi-l^2\nabla^2\Psi)_{,x},\\
2l^2\nabla^2\Phi_{,x} &= (\Psi-l^2\nabla^2\Psi)_{,y}
\end{aligned}\tag{1.18}$$

or in the separated form

$$\nabla^4\Phi = 0, \qquad \nabla^2\Psi-l^2\nabla^4\Psi = 0. \tag{1.19}$$

2. LATTICE ANALOGY

Consider a regular, orthogonal two-dimensional lattice consisting of elastic beams parallel to the axes x, y of a rectangular coordinate system (Fig. 1). The dimensions of an elementary rectangle are $2a \times 2b$, and the cross-sectional areas of the corresponding beams, their moments of inertia and the Young's moduli are A_x, E_x, I_x and A_y, E_y, I_y, respectively.

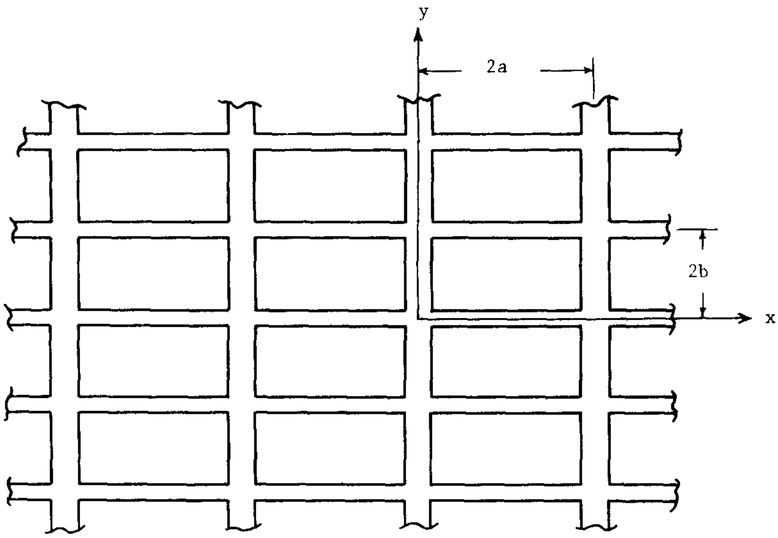


FIG. 1

Consider first the state of equilibrium of an elementary cross-shaped framework shown in Fig. 2. Denote by N_x, N_y—the normal forces, by V_{xy} and V_{yx}—the shearing forces, and by M_x and M_y—the bending moments acting in the horizontal and vertical members of the element at its center. If the increments of these forces along the arms of the elementary cross are denoted by symbols Δ, the three conditions of equilibrium of the element shown in Fig. 2 are

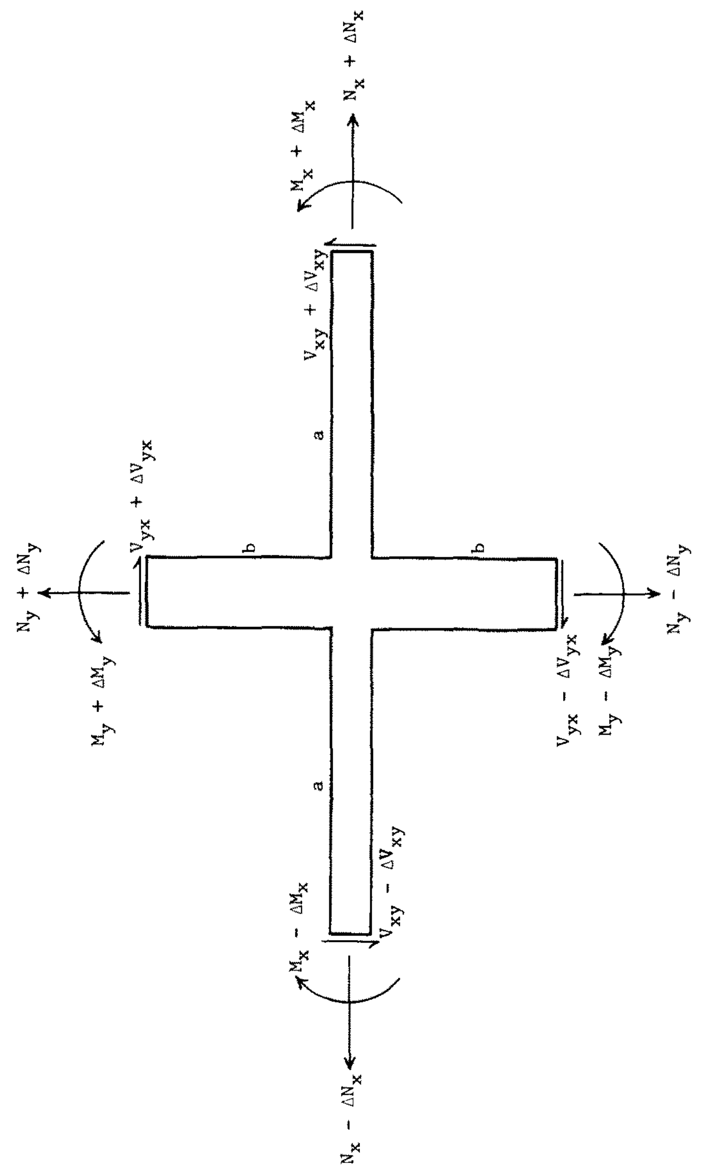
$$\begin{aligned} \Delta N_x + \Delta V_{yx} &= 0, \\ \Delta N_y + \Delta V_{xy} &= 0, \\ \Delta M_x + \Delta M_y - bV_{yx} + aV_{xy} &= 0. \end{aligned} \tag{2.1}$$

For a reasonably smooth variation of the forces across the element, the increments ΔN, ΔV, ΔM can be approximately replaced by the first terms of the Taylor power series expansions of functions N, V, M; for instance

$$\Delta N_x \approx \frac{\partial N_x}{\partial x} a \quad \text{etc.} \tag{2.2}$$

When the expressions (2.2) are substituted into equations (2.1) and each of these equations is divided by $2ab$, the equilibrium conditions of the element shown in Fig. 2 can be written in the form

$$\begin{aligned} \bar{\sigma}_{xx,x} + \bar{\sigma}_{yx,y} &= 0, \\ \bar{\sigma}_{xy,x} + \bar{\sigma}_{yy,y} &= 0, \\ \bar{m}_{xz,x} + \bar{m}_{yz,y} - \bar{\sigma}_{yx} + \bar{\sigma}_{xy} &= 0, \end{aligned} \tag{2.3}$$



Nx + ΔNx
Mx + ΔMx
Vxy + ΔVxy
a
Vyx + ΔVyx
Ny + ΔNy
My + ΔMy
b
b
Ny - ΔNy
Vyx - ΔVyx
My - ΔMy
a
Vxy - ΔVxy
Mx - ΔMx
Nx - ΔNx

FIG. 2

where $\bar{\sigma}$ and $\bar{m}$ denote the reduced stresses in the lattice element

$$\bar{\sigma}_{xx} = \frac{N_x}{2b}, \qquad \bar{\sigma}_{yy} = \frac{N_y}{2a},$$
$$\bar{\sigma}_{xy} = \frac{V_{xy}}{2b}, \qquad \bar{\sigma}_{yx} = \frac{V_{yx}}{2a}, \tag{2.4}$$
$$\bar{m}_{xz} = \frac{M_x}{2b}, \qquad \bar{m}_{yz} = \frac{M_y}{2a}.$$

The reduced stresses can be decomposed into the symmetric and antisymmetric parts

$$\bar{\sigma}_{xy} = \bar{s}_{xy} + \bar{r}_{xy}, \qquad \bar{\sigma}_{yx} = \bar{s}_{xy} + \bar{r}_{yx},$$
$$\bar{s}_{xy} = \bar{s}_{yx} = \tfrac{1}{2}(\bar{\sigma}_{xy} + \bar{\sigma}_{yx}), \tag{2.5}$$
$$\bar{r}_{xy} = -\bar{r}_{yx} = \tfrac{1}{2}(\bar{\sigma}_{xy} - \bar{\sigma}_{yx}).$$

The antisymmetric part $\bar{r}_{xy}$ of $\bar{\sigma}_{xy}$ can be eliminated from equations (2.3) to yield the equilibrium conditions in the form

$$\bar{s}_{xx,x} + \bar{s}_{xy,y} + \tfrac{1}{2}(\bar{m}_{yz,y} + \bar{m}_{xz,x})_{,y} = 0,$$
$$\bar{s}_{xy,x} + \bar{s}_{yy,y} - \tfrac{1}{2}(\bar{m}_{yz,y} + \bar{m}_{xz,x})_{,x} = 0,$$

completely analogous to conditions (1.4) derived for a continuous medium element in Section 1.

To complete this analogy, the reduced strains $\bar{\gamma}$ and $\bar{\varkappa}$ have to be defined. The displacements and slopes of a lattice element (Fig. 3) can be calculated by applying the elementary formulas of strength of materials and by assuming that the element is composed of four cantilever beams loaded by axial forces N, transversal forces V and bending moments M.

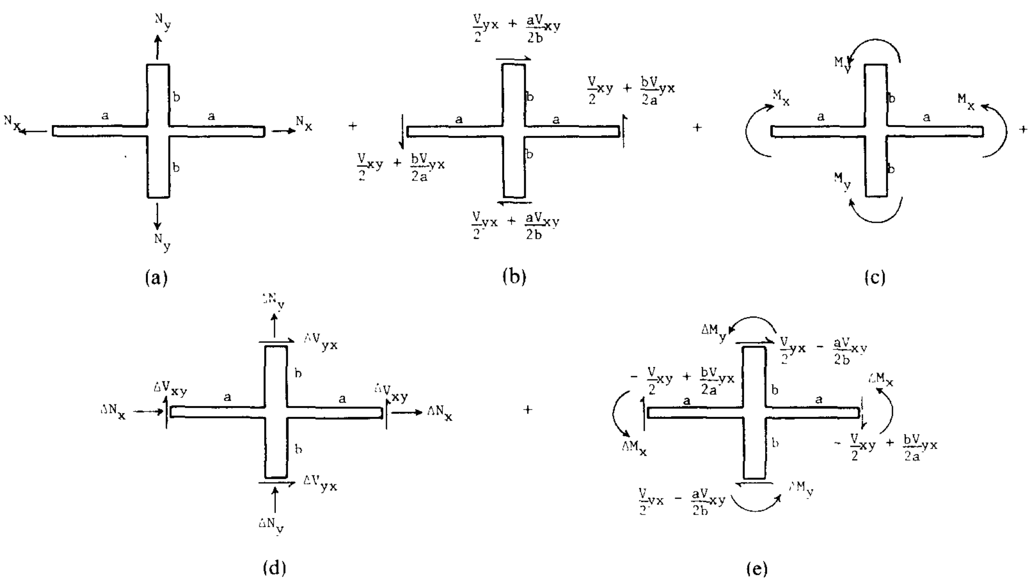


FIG. 3

If the influence of the incremental force systems shown in Fig. 3d, e is neglected, the reduced strains can be defined in the following way.

(a) Reduced normal strain is the average longitudinal strain of the rod (Fig. 3a)

$$\bar{\gamma}_{xx} = \frac{N_x}{E_x A_x}, \qquad \bar{\gamma}_{yy} = \frac{N_y}{E_y A_y} \tag{2.6}$$

(b) Reduced shear strain is the average value of transversal deflection per unit length of the perpendicular beams

$$\bar{\gamma}_{xy} = \frac{1}{2}\left(\frac{\delta_{xy}}{a} + \frac{\delta_{yx}}{b}\right). \tag{2.7}$$

Here δ_{xy}, δ_{yx} are the respective deflections caused by V_{xy}, V_{yx}

$$\delta_{xy} = \frac{1}{2}\left(V_{xy} + \frac{b}{a} V_{yx}\right)\frac{a^3}{3E_x I_x},$$

$$\delta_{yx} = \frac{1}{2}\left(\frac{a}{b} V_{xy} + V_{yx}\right)\frac{b^3}{3E_y I_y}.$$

(c) Reduced torsion-flexure strain is the average change of slope per unit length of the beam (average curvature)

$$\bar{\varkappa}_{xz} = \frac{M_x}{E_x I_x}, \qquad \bar{\varkappa}_{yz} = \frac{M_y}{E_y I_y}. \tag{2.8}$$

Now, the forces N, V, M appearing in equations (2.6), (2.7) and (2.8) are replaced by the previously introduced reduced stresses (2.4) and (2.5). The resulting equations, which may serve as reduced stress–strain relations for an orthogonal lattice, can be written in the form

$$\begin{aligned} \bar{s}_{xx} &= \bar{E}_x \bar{\gamma}_{xx}, \qquad \bar{s}_{yy} = \bar{E}_y \bar{\gamma}_{yy}, \qquad \bar{s}_{xy} = 2\bar{G}\bar{\gamma}_{xy} \\ \bar{m}_{xz} &= 4\bar{G} l_x^2 \bar{\varkappa}_{xz}, \qquad m_{yz} = 4\bar{G} l_y^2 \bar{\varkappa}_{yz} \end{aligned} \tag{2.9}$$

in which the constants $\bar{E}$, $\bar{G}$ and l are expressed in terms of the mechanical and geometric characteristics of the lattice elements as follows:

$$\begin{aligned} \bar{E}_x &= \frac{E_x A_x}{2b}, \qquad \bar{E}_y = \frac{E_y A_y}{2a}, \\ \bar{G} &= \frac{3E_x E_y I_x I_y}{2ab(aE_y I_y + bE_x I_x)}, \\ l_x &= \frac{a}{\sqrt{6}}\left(\frac{1}{2} + \frac{b}{2a}\frac{E_x I_x}{E_y I_y}\right), \\ l_y &= \frac{b}{\sqrt{6}}\left(\frac{1}{2} + \frac{a}{2b}\frac{E_y I_y}{E_x I_x}\right). \end{aligned} \tag{2.10}$$

Equations (2.9) are identical with the stress–strain relations (1.6) introduced in Section 1 for a continuous orthotropic medium. This confirms the assumed analogy and proves than an orthogonal lattice may serve as a good illustration of phenomena occurring in Cosserat-type continua. Likewise, certain solutions known from the theory of couple-stresses can serve as approximate solutions for analogous problems concerning two-dimensional gridworks. In cases when the lattice structure contains a very large number of elements it can prove to be advantageous to replace the discrete structure by a continuous Cosserat-type continuum; the analytical solution of the associated couple-stress problem can be much simpler than the numerical solution of the original set of linear algebraic equations which results from the usual elementary structural mechanics approach to the problem.

In connection with the formulae (2.10) it can be observed that in the case of a square net $a \times a$ of identical beams, the associated continuum model does not become isotropic. This is due to the internal structure of the lattice, and even in the fixed coordinate system x, y it is seen that for isotropic elastic bodies the relation $E = 2G(1+\nu)$ holds, whereas in the case of a lattice, with $\nu = 0$, $\bar{E} = EA/2a$, $\bar{G} = 3EI/4a^3$ and hence, generally $\bar{E} \neq 2\bar{G}$.

The scond observation concerns the magnitude of the additional elastic constant $\bar{l}$. In the case of a square net and equal beam rigidities it follows from equation (2.10) that

$$\bar{l}_x = \bar{l}_y = a/\sqrt{6} = 0{\cdot}408a$$

which sheds some additional light on the order of magnitude of l in real bodies characterized by a certain crystalline or granular microstructure.

3. TWO PARTICULAR CASES

The stress functions Φ and Ψ, derived in Section 1 and originally introduced by Mindlin [2] are employed to obtain the solution of two particular couple-stress problems. The first problem concerns the influence of couple-stresses on the stress-concentration around a perfectly rigid circular inclusion in an infinite isotropic plane subject to uniform compression in the direction of the x-axis (Fig. 4).

All the general formulae derived in Section 1 are transformed to polar coordinates $r = (x^2+y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1} y/x$, and the two functions $\Phi(r, \theta)$ and $\Psi(r, \theta)$ must satisfy the equations

$$\begin{aligned} (\Psi - l^2\nabla^2\Psi)_{,r} &= -2(1-\nu)l^2\frac{1}{r}\nabla^2\Phi_{,\theta}, \\ \frac{1}{r}(\Psi - l^2\nabla^2\Psi)_{,\theta} &= 2(1-\nu)l^2\nabla^2\Phi_{,r}, \end{aligned} \tag{3.1}$$

and the following boundary conditions

(a) Perfect clamping of the plate at the boundary of the inclusion

$$u_r(r, \theta) = u_\theta(r, \theta) = \frac{\partial u_\theta(r, \theta)}{\partial r} = 0 \quad \text{for} \quad r = a; \tag{3.2}$$

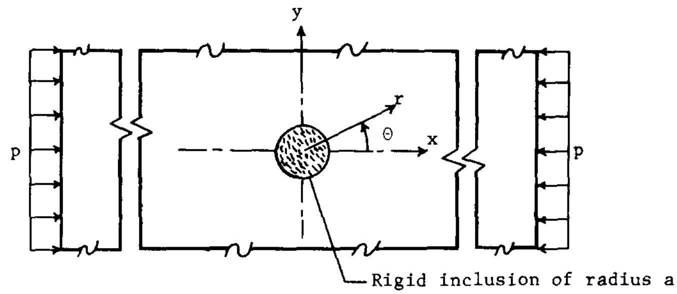


FIG. 4

(b) Uniform compression in the x-direction at infinity, therefore for $r \to \infty$

$$\begin{aligned} s_{rr} &= -\frac{p}{2}(1+\cos 2\theta), \\ s_{\theta\theta} &= -\frac{p}{2}(1-\cos 2\theta), \\ s_{r\theta} &= \frac{p}{2}\sin 2\theta. \end{aligned} \tag{3.3}$$

The stresses, in terms of functions Φ and Ψ, can now be written

$$\begin{aligned} s_{rr} &= \frac{1}{r}\Phi_{,r}+\frac{1}{r^2}\Phi_{,\theta\theta}-\frac{1}{r}\Psi_{,r\theta}+\frac{1}{r^2}\Psi_{,\theta}, \\ s_{\theta\theta} &= \Phi_{,rr}+\frac{1}{r}\Psi_{,r\theta}-\frac{1}{r^2}\Psi_{,\theta}, \\ s_{r\theta} &= -\frac{1}{r}\Phi_{,r\theta}+\frac{1}{r^2}\Phi_{,\theta}+\frac{1}{2}\left(\Psi_{,rr}-\frac{1}{r}\Psi_{,r}-\frac{1}{r^2}\Psi_{,\theta\theta}\right), \\ m_{rz} &= \Psi_{,r}, \qquad m_{\theta z} = \frac{1}{r}\Psi_{,\theta} \end{aligned} \tag{3.4}$$

The suitable form of the stress functions for solving the problem under consideration is

$$\Phi(r,\theta) = -\frac{pr^2}{4}(1-\cos 2\theta)+C_1\ln r+\left(\frac{C_2}{r^2}+C_3\right)\cos 2\theta,$$

$$\Psi(r,\theta) = \left[\frac{C_4}{r^2}+C_5K_2\left(\frac{r}{l}\right)+C_6I_2\left(\frac{r}{l}\right)\right]\sin 2\theta.$$

Here I_n and K_n denote the modified Bessel functions of first and second kind and order n, and $C_1,\ldots,C_6$ are constants of integration. These constants are determined from the boundary conditions (3.2), (3.3),

$$\begin{aligned}
C_1 &= -a^2(1-2\nu)\frac{p}{2},\\
C_2 &= -\frac{a}{F}\left[(3-2\nu)+\frac{\alpha K_0(\alpha)}{2K_1(\alpha)}\right]\frac{p}{2},\\
C_3 &= \frac{a}{F}\left[2+\frac{\alpha K_0(\alpha)}{K_1(\alpha)}\right]\frac{p}{2},\\
C_4 &= \frac{8(1-\nu)a^4}{\alpha^2 F}\left[2+\frac{\alpha K_0(\alpha)}{K_1(\alpha)}\right]\frac{p}{2},\\
C_5 &= -\frac{8(1-\nu)a^2 K_0(\alpha)}{\alpha F K_0(\alpha) K_1(\alpha)},\\
C_6 &= 0,
\end{aligned} \tag{3.5}$$

in which the notations

$$\alpha = \frac{a}{l}, \qquad F = 2(1-2\nu)+(3-4\nu)\alpha K_0(\alpha)/K_1(\alpha)$$

were employed.

The maximum value of normal stress occurs at the ends of the horizontal diameter of the inclusion, $r = a$, $\theta = 0$ or π.

$$\sigma_{rr}^{\max} = -\frac{p}{2}\left[(3-2\nu)+\frac{2(3-2\nu)+\alpha K_0(\alpha)/K_1(\alpha)}{2(1-2\nu)+(3-4\nu)\alpha K_0(\alpha)/K_1(\alpha)}\right].$$

This formula, when compared with the corresponding classical result by Goodier [3]

$$\sigma_{rr}^{\max} = -\frac{p}{2}\frac{2(5-4\nu)(1-\nu)}{3-4\nu}$$

shows that couple-stresses result in an increase of the stress concentration factor (see Fig. 5). If it is assumed, for instance, that $\nu = \frac{1}{4}$, the classical theory gives the result $k = \sigma_{rr}^{\max}/p = 1{\cdot}5$, whereas the couple-stress theory leads to $k = 3{\cdot}75$. This increase is contrary to the results obtained by Mindlin [2], who found a decrease in the stress concentration factor for the case of a circular hole in an infinite plane under compression. On the other hand, a similar increase of stress concentration factors was observed by Muki and Sternberg [4] and Day and Weitsman [5] at the contact surface between two bodies composed of different materials. As could be expected, the influence of couple-stresses decreases rapidly with increasing distance from the boundary of the inclusion.

The second example concerns the problem of an infinite elastic plane loaded by a torque T applied to the boundary of a circular hole of radius a. Two possibilities are considered: in the first case the load consists of uniformly distributed tangential forces $t = T/2\pi a^2$; this leads to the well-known, elementary solution. In the second case, uniformly distributed couples $m_r = T/2\pi a$ act along the circular boundary, and the solution is found with the aid of the couple-stress theory.

The stress potentials are assumed in the same form for both cases

$$\Phi = C_0\theta,$$
$$\Psi = C_1K_0(r/l)+C_2I_0(r/l).$$

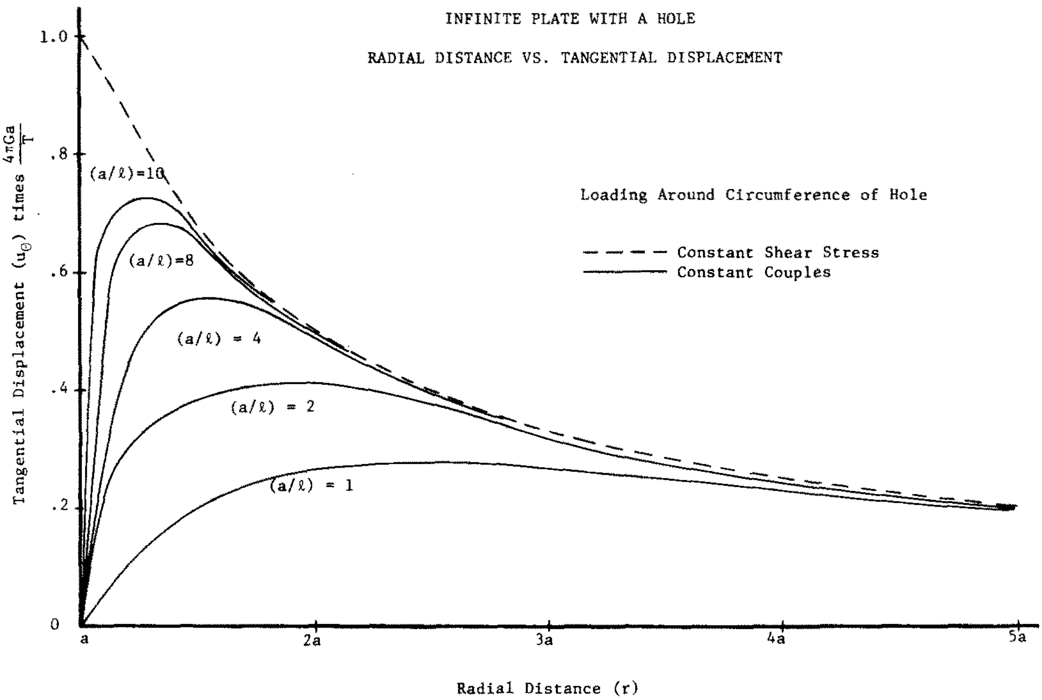


FIG. 5

Since all stresses, in both problems, must vanish as $r \to \infty$ it follows that $C_2 = 0$. The remaining boundary conditions for the case of uniformly distributed shears are

$$\sigma_{r\theta}(a, \theta) = T/2\pi a^2, \qquad m_{rz}(a, \theta) = 0, \qquad \sigma_{rr}(a, \theta) = 0,$$

and for the case of uniformly distributed couples are

$$\sigma_{r\theta}(a, \theta) = 0, \qquad m_{rz}(a, \theta) = T/2\pi a, \qquad \sigma_{rr}(a, \theta) = 0.$$

Therefore, the non-zero stresses and displacements can be written
(a) in the case of uniformly distributed shear (classical continuum) as

$$\begin{aligned} \sigma_{r\theta} &= T/2\pi r^2, \\ u_\theta &= -T/4\pi Gr; \end{aligned} \tag{3.6}$$

(b) in the case of uniformly distributed couples (Cosserat-type continuum) as

$$
\begin{aligned}
\sigma_{r\theta} &= \frac{T}{2\pi r^2}\left[1-\frac{r}{a}\frac{K_1(r/l)}{K_1(\alpha)}\right],\\
\sigma_{\theta r} &= \frac{T}{2\pi r^2}\left[1-\frac{r}{a}\frac{K_1(r/l)}{K_1(\alpha)}-\frac{r^2}{la^2}\frac{K_0(r/l)}{K_1(\alpha)}\right],\\
m_{rz} &= -\frac{T}{2\pi a}\frac{K_1(r/l)}{K_1(\alpha)},\\
u_\theta &= -\frac{T}{4\pi Gr}\left[1-\frac{r}{a}\frac{K_1(r/l)}{K_1(\alpha)}\right].
\end{aligned}
\tag{3.7}
$$

The first result of interest involves the limiting case $l \to 0$ (the transition from the Cosserat continuum to the classical elastic material). It is noted that the solution for the second case then becomes identical with the solution of the first case for any $r > a$. Hence, for classical type continuum the solution for $r > a$ is independent of how the torque is applied. The result is in marked contrast with what would be expected for Cosserat-type continua. This difference is most easily vizualized by replacing the Cosserat continuum by a lattice of the type previously discussed in Section 2 of this paper. Now, it is obvious that the lattice will respond quite differently depending upon whether it is subject to shear

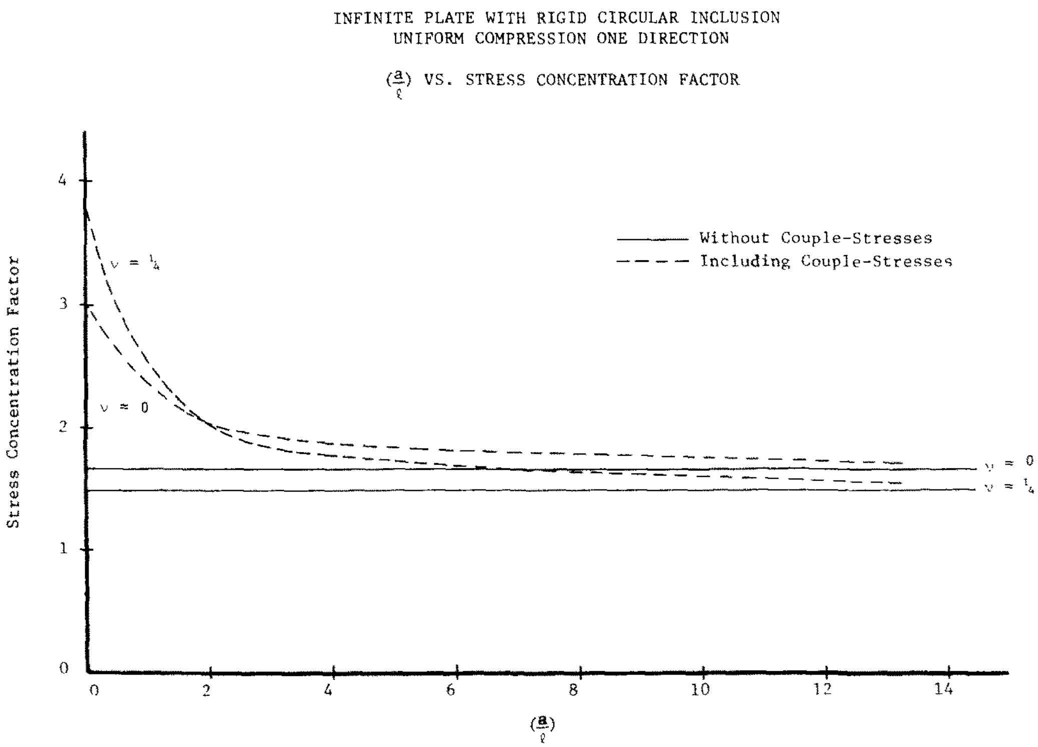


FIG. 6

forces or bending moments and thus, the solutions would be quite different. These two solutions also illustrate the difference between the force transmitting mechanisms of the two types of continua.

A second conclusion can be drawn from the behavior of solutions (2.6) and (2.7) under the assumption that $0 < l \ll a$ and $r \gg a$; then the Bessel functions can be represented by the first terms of their asymptotic expansions

$$\frac{K_n(r/l)}{K_m(a/l)} \sim \left(\sqrt{\frac{a}{r}}\right)\exp -\frac{a}{l}\left(\frac{r}{a}-1\right).$$

It now follows that retention of the first term of each of equations (3.6) is all that is required for adequate representation of stresses and displacements for large values of r. Hence, the solution for the case of uniformly distributed couples in Cosserat-type continua becomes asymptotic to the uniformly distributed shears problem in a classical medium for $r \gg a$; see Fig. 6 for a comparative plot of the displacements for the two cases. This comparison also further strengthens the previous observation that the effects of couple-stresses are most prominent in the immediate vicinity of the boundary.

For small values of a/l the above argument does not hold and the deviations between the two solutions are quite noticeable.

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Абстракт—Рассматривается двухмерная задача равновесия идеально упругого тела с моментными напряжениями. В части 1 дается общая форма решения задачи для случая ортотропной среды. Часть 2 указывает на тесную аналогию между уравнениями, описывающими поведение плоской прямоугольной решетки, состоящей из жестких, соединенных упругих балок и общей системой уравнений двухмерной теории моментных уравнений для некоторых ортотропных тел. Часть 3 заключает две задачи, которые указуют на эффекты моментных напряжений и их поведение на напряжения и перемещения в окрестности кругогой инклюзии и кругого отверствия в бесконечной упругой плоскости.